Standing Waves on a Hanging Rope

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The problem of standing waves on a uniform rope or chain of negligible stiffness hanging vertically in the gravitational field has been considered by a number of authors.[1–5] When the rope or chain is driven to oscillate in a plane, the solution leads to Bessel functions of the first order,[1,5] which allows the determination of the standing-wave frequencies and nodal positions. While good agreement with experiment has been obtained, the treatment is beyond the scope of introductory courses in physics.

The equations for a rope or chain whirling about a vertical axis have solutions similar to those for the same rope or chain swinging in a vertical plane, provided the amplitudes of oscillation are small, and it is much easier to set up the whirling motions experimentally. Figure 1 shows a hanging chain in its first three standing wave modes, generated by the whirling method. These photos appeared first in a text by French,[6] and later in a text by Halliday and Resnick.[7] Western[5] presents photos of modes 2, 3, and 4. The nodes are not placed as they would be for uniform velocity of the wave. Most dramatically, in the higher modes the displacement nodes get farther apart as you go up the chain.

This paper will present a second approach to the problem of a uniform rope oscillating in a vertical plane using methods from first-year physics courses. While not as numerically accurate as the former approach, it displays the physical concepts in a particularly clear way.

First we review the work of Satterly[1] contained in Eqs. (1) to (4) below. The speed (v) of a wave pulse in a rope of tension τ and mass per unit length μ is given by the standard equation:

$$v = \sqrt{\frac{\tau}{\mu}} \qquad (1)$$

The tension in a stationary, hanging rope a distance x up from the bottom is

$$\tau = mg\frac{x}{L} = \mu g x \qquad (2)$$

where L is the total length of the rope, τ is just the weight of the rope below x, and $\frac{m}{L} = \mu$.

Eliminating the tension between Eq. (1) and Eq. (2) and solving for the velocity, we get

$$v = \sqrt{gx} \qquad (3)$$

A standard equation in accelerated motion is $v = \sqrt{2ax}$, where x is the distance traveled while the velocity changed from 0 to v. Comparing this with Eq. (3), we see that $2a = g$ or $a = \frac{g}{2}$. So the wave accelerates is beyond the scope of introductory courses in physics.

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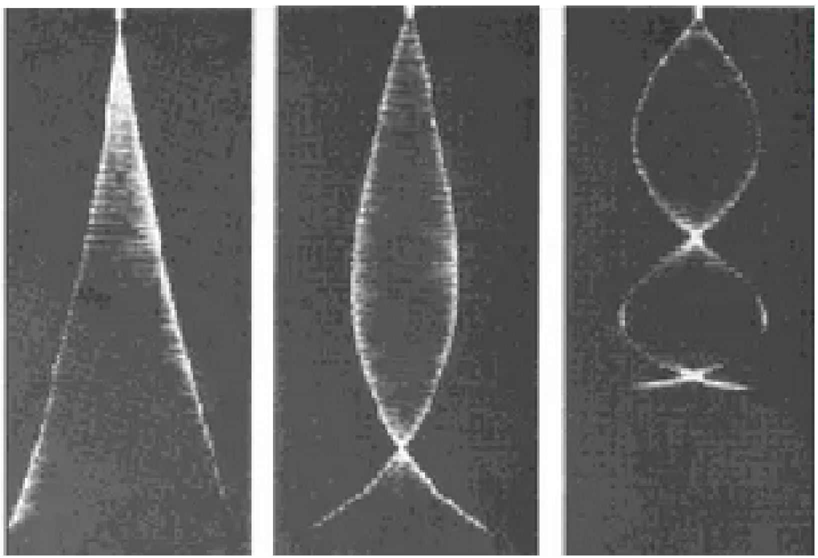

Fig. 1. Chain in its first three standing-wave modes.

Photos by Jon Rosenfeld, *Education Research Center, MIT.*

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A standard equation in accelerated motion is $v = \sqrt{2ax}$, where x is the distance traveled while the velocity changed from 0 to v. Comparing this with Eq. (3), we see that $2a = g$ or $a = \frac{g}{2}$. So the wave accelerates upward (the direction of increasing x) with acceleration $\frac{g}{2}$. The wave speeds up as it ascends the rope, and slows down as it descends.

It is now clear why the nodes in Fig. 1 are not evenly spaced. The farther up the rope the wave pulse goes, the greater the tension and the greater the wave velocity. Wave pulses moving up the rope will stretch out, since the leading edge of the pulse will be moving faster than the trailing edge.

The time for the wave pulse to go from the bottom to the top of the rope is gotten from the standard equation for accelerated motion:

$$L = \frac{1}{2} a\, t^2 = \frac{1}{2}\left(\frac{g}{2}\right) t^2 = \frac{1}{4} g\, t^2$$

So,

$$t = 2\sqrt{\frac{L}{g}} \qquad (4)$$

Satterly[1] presents experimental data in agreement with Eq. (4). Freeman[4] does not mention the constant value of the acceleration. He derives Eq. (4) from Eq. (3) by separating variables and integrating. A problem based on the Freeman paper has appeared in recent editions of the texts by Halliday and Resnick, in which the student is asked to derive Eqs. (3) and (4).

We are now in position to calculate the standing-wave frequencies and the nodal positions for any mode of the rope. The usual relation between velocity, wavelength (λ) and frequency (f) is

$$v = \lambda f \qquad (5)$$

For the hanging rope, velocity depends on position on the rope, so wavelength is not a defined quantity. However, in any standing wave mode, the frequency will be the same for all positions along the rope. If this were not so, different positions on the rope would have different frequencies, and the standing wave would not be maintained.

To get the first standing-wave mode, set the time in Eq. (4) equal to $\frac{T_1}{4}$, where T_1 is the period of the first standing-wave mode. This is equivalent to equating a quarter period to the distance between a node and an antinode, and half a period to the distance between adjacent nodes.) Now

Table I. Model predictions for frequencies of first five standing-wave modes. Each number in table should be multiplied by $\sqrt{\frac{g}{L}}$ to give frequency in hertz.

Mode	Gibbs	Satterly[1]
1	0.125	0.191
2	0.375	0.439
3	0.625	0.689
4	0.875	0.939
5	1.125	1.190

$$f_2 = \frac{3}{8}\sqrt{\frac{g}{L}} = 3f_1 \qquad (7)$$

Since the node is $\frac{1}{3}$ of the way up the rope in time, it must be $(\frac{1}{3})^2 = \frac{1}{9}$ of the way up in distance, since x is proportional to t^2 under constant acceleration conditions.

For the third mode, set the time in Eq. (4) equal to $\frac{5}{4}T_3$, corresponding to one quarter period from the bottom to the first node, and a half period between each pair of nodes. This gives

$$f_3 = \frac{5}{8}\sqrt{\frac{g}{L}} = 5f_1 \qquad (8)$$

The lower node is $\frac{1}{5}$ of the way up in time or $\frac{1}{25}$ of the way up in dis-

from the bottom by n, the nth node in the mth standing wave mode appears at

$$\frac{(2n-1)^2}{(2m-1)^2}L \text{ from the bottom}$$

of the rope. ($n = 1, 2, 3, 4, \ldots$ and $m = 1, 2, 3, 4, \ldots\ n \leq m$.) The nodal positions predicted by this model agree qualitatively with the photographs of Fig. 1. However, the distance of the first node from the bottom is noticeably greater than predicted. For example, the first node in the second mode is closer to one-fifth of the way up from the bottom than one-ninth.

Table I compares the standing-wave frequencies for the first five modes predicted by this model with those predicted by Satterly.[1] The numbers given in Table I should be multiplied by $\sqrt{\frac{g}{L}}$ to give the frequencies in hertz. Two things are immediately evident. First, the frequencies predicted in this model are smaller than those given by Satterly. Second, the Satterly model does not lead to a harmonic series.

Calculus provides an alternative way to derive the acceleration from Eq. (3). From the chain rule we can write:

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v = \frac{g}{2}$$

$$v = \lambda f \quad (5)$$

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To get the first standing-wave mode, set the time in Eq. (4) equal to $\frac{T_1}{4}$, where T_1 is the period of the first standing-wave mode. This is equivalent to saying that a wave started at the top must go to the bottom and back to the top in half a period, arriving out of phase with the next wave to be generated by the driver, thus producing a node at the top by destructive interference. Then $\frac{T_1}{4} = 2\sqrt{\frac{L}{g}}$ and $T_1 = 8\sqrt{\frac{L}{g}}$. Thus, the frequency of the first mode is

$$f_1 = \frac{1}{T_1} = \frac{1}{8}\sqrt{\frac{g}{L}} \quad (6)$$

To get the frequency of the next higher mode, set the time in Eq. (4) equal to $\frac{3}{4}T_2$. This corresponds to $\frac{1}{4}T_2$ for the length below the first node and $\frac{1}{2}T_2$ for the distance from the first node to the top of the rope. (In general we assign one quarter

the rope in time, it must be $(\frac{1}{3})^2 = \frac{1}{9}$ of the way up in distance, since x is proportional to t^2 under constant acceleration conditions.

For the third mode, set the time in Eq. (4) equal to $\frac{5}{4}T_3$, corresponding to one quarter period from the bottom to the first node, and a half period between each pair of nodes. This gives

$$f_3 = \frac{5}{8}\sqrt{\frac{g}{L}} = 5f_1 \quad (8)$$

The lower node is $\frac{1}{5}$ of the way up in time or $\frac{1}{25}$ of the way up in distance. The upper node is $\frac{3}{5}$ of the way up in time or $\frac{9}{25}$ of the way up in distance.

For each higher mode, we decrease the period by adding one half period to the (constant) travel time for each additional node. It is clear from Eqs. (6), (7), and (8) that in this approximation, the standing wave frequencies form a series of odd harmonics. This is what we might expect for a medium free at one end and fixed at the other.

For the m^{th} mode, $f_m = \frac{2m-1}{8}\sqrt{\frac{g}{L}} = (2m-1)f_1$, and the nodes appear at $\frac{L}{(2m-1)^2}$, $\frac{9L}{(2m-1)^2}$, $\frac{25L}{(2m-1)^2}$, etc., from the bottom. If we number the nodes up

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$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v = \frac{g}{2} \quad (9)$$

Using this method allows us to generalize the solution[8] to a linear mass density of form $\mu = kx^n$. To find the tension in the rope a distance x from the bottom, we integrate to find the total weight of rope below x. Letting the integration variable be x', we write

$$\tau = \int_0^x \mu g dx' = \int_0^x k(x')^n g dx' = \frac{kgx^{n+1}}{n+1} \quad \text{for } n \geq 0. \quad (10)$$

We now obtain the velocity by Eq. (1):

$$v = \sqrt{\frac{gx}{n+1}} \quad (11)$$

The velocity has the same dependence on x as in Eq. (3) for the uniform rope, leading to a constant acceleration $a = \frac{g}{2(n+1)}$. The frequency values change due to the factor $(n + 1)$ in the acceleration, but since the acceleration is still constant, the nodal positions do not change, and a series of odd harmonics is retained.

I find this model pedagogically interesting for a number of reasons. (1) It provides students with a new opportunity to apply the constant-acceleration equations. (2) It emphasizes that the condition for a standing wave is based on time, not length. Thus for the first harmonic of the rope, we state that the wave must travel the length of the rope in $T_1/4$. By contrast, for constant-velocity waves, we usually state that the first standing wave occurs when the rope is one-quarter wavelength long. Of course in the case of the vertical rope, the wavelength is not a defined quantity, but the condition on travel time is valid for both constant-velocity and constant-acceleration cases, and is therefore the more general condition. (3) The standing-wave frequencies of the rope form a series of odd harmonics, in spite of the acceleration of the wave. This illustrates the power of the boundary conditions in determining the standing-wave frequencies. (4) It demonstrates that not all physical models are highly accurate.

Both models assume Eq. (2) holds. But this is the static condition for the tension in the rope, and it can only be an approximation to the true tension under dynamic conditions. In addition, the Satterly[1] approach assumes that the displacements of the rope are small, which is clearly violated by the chain in Fig. 1. In the model presented here, Eq. (1) is also an approximate relation. This result is usually derived under conditions in which the tension of the medium is much greater than its weight, which is not the case for the hanging rope.

Acknowledgment

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