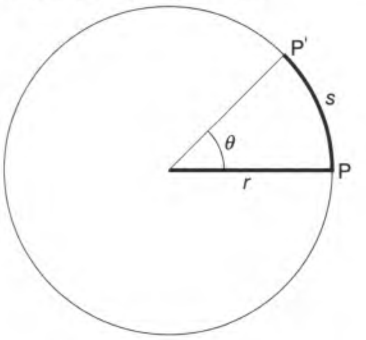
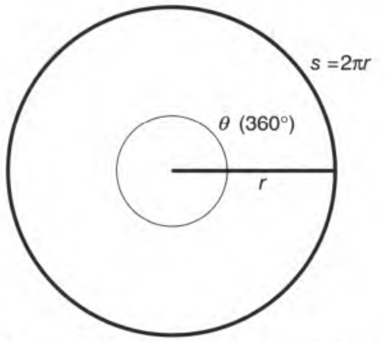
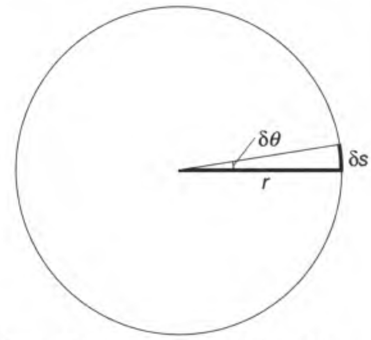
B14 Circular motion

Angular displacement







If point P moves to P′, then the angle θ is called the ***angular displacement***. It is measured in ***radians*** (see also A1):

$$\theta = \frac{s}{r}$$

If s is the full circumference of the circle,

$$\theta = \frac{2\pi r}{r} = 2\pi$$

So 2π radians = 360°

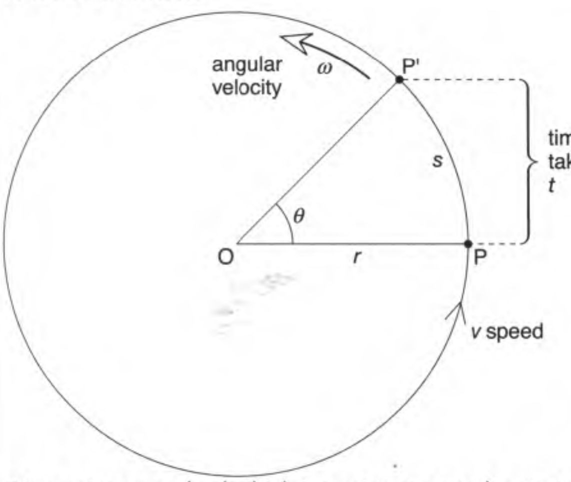
$$\therefore \ 1 \text{ radian} = \frac{360}{2\pi} = 57.3°$$

Above, $\delta\theta$ is a very small angle. ($\delta\theta$ counts as one symbol.) δs is so small that it can either be the arc of a circle or the side of a triangle. So

$$\sin \delta\theta = \frac{s}{r} = \delta\theta$$

i.e. for *small* angles $\sin \delta\theta = \delta\theta$.

Rate of rotation



P is a point on a wheel which is turning at a steady rate. In time t, it moves to P′. The rate of rotation can be measured either as an ***angular velocity*** or as a ***frequency***.

Angular velocity ω

$$\text{angular velocity} = \frac{\text{angular displacement}}{\text{time taken}}$$

In symbols $\omega = \frac{\theta}{t}$

For example, if a wheel turns through 10 radians in 2 seconds, then $\omega = 5$ rad s^{-1}. Angular frequency is also measured in rad s^{-1}. It is the magnitude of the vector angular velocity.

Frequency f

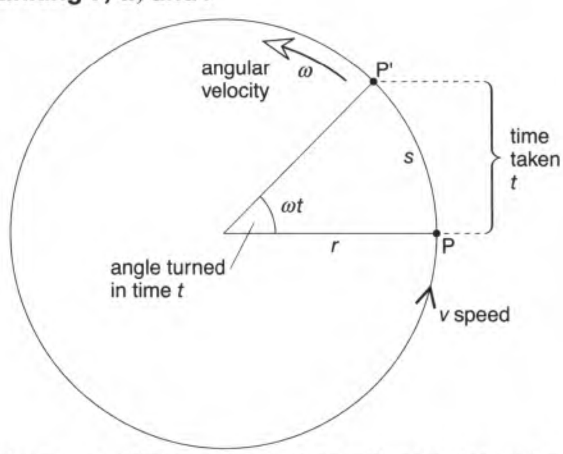
$$\text{frequency} = \frac{\text{number of rotations}}{\text{time taken}}$$

Frequency is measured in hertz (Hz). For example, if a wheel completes 12 rotations in 4 seconds, then $f = 3$ Hz.

Period T This is time taken for one rotation. If a wheel makes 3 complete rotations per second ($f = 3$ Hz), then the time taken for one rotation is $\frac{1}{3}$ second. So

$$T = \frac{1}{f}$$

Linking v, ω, and r



Above, a particle is moving in circle with a steady speed v. (It is not a steady velocity because the direction of the velocity vector is changing.) The particle moves a distance s in time t, so

$$v = \frac{s}{t}$$

As the angular velocity is ω, the angle turned in time t is ωt (from the equation on the left). But $\omega t = s/r$. So $s = \omega tr$. Substituting this in the above equation gives

$$v = \omega r$$

Linking ω, f, and T As there are 2π radians in one full rotation (360°),

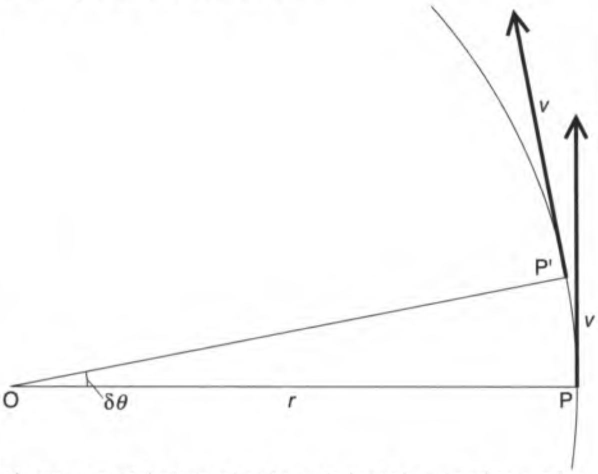
$$\omega = 2\pi f$$

For example, a wheel turning at 3 rotations per second ($f = 3$ Hz) has an angular velocity of 6π radians per second.

As $T = 1/f$, it follows from the previous equation that

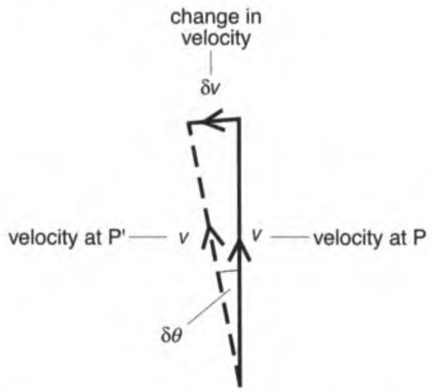
$$T = \frac{2\pi}{\omega}$$

Centripetal acceleration



Above, a particle is moving in a circle with a steady speed v. The diagram shows how the velocity vector changes direction as the particle moves from P to P′ in time δt.

Below, the velocity vectors from the previous diagram have been used in a triangle of vectors (see B4).



The δv vector represents the *change* in velocity because it is the velocity vector which must be *added* to the velocity at P to produce the new velocity (the resultant) at P′. Note that the change in velocity is towards O. In other words, the particle has an *acceleration* towards the centre of the circle. This is called ***centripetal acceleration***.

If a is the centripetal acceleration, $a = \frac{\delta v}{\delta t}$

But, from the triangle above, $\delta\theta = \frac{\delta v}{v}$. So $\delta v = v\delta\theta$

Substituting this in the previous equation, $a = \frac{v\delta\theta}{\delta t}$

But $\delta\theta = \omega\delta t$. So $a = v\omega$

Using $v = \omega r$, two more versions of the above equation can be obtained. So

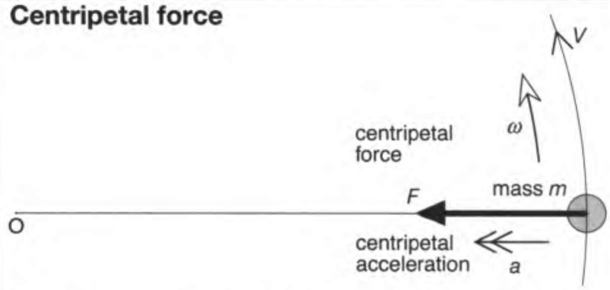
$a = v\omega$ $a = \frac{v^2}{r}$ $a = \omega^2 r$

For example, if a particle is moving at a steady speed of 3 m s^{-1} in a circle of radius 2 m, its centripetal acceleration a is found using the middle equation: $a = 3^2/2 = 4.5$ m s^{-2}.

Note:

- When something accelerates, its velocity changes. As velocity is a vector, this can mean a change in *speed* or *direction* (or both). Centripetal acceleration is produced by a change in direction, not speed.

Centripetal force

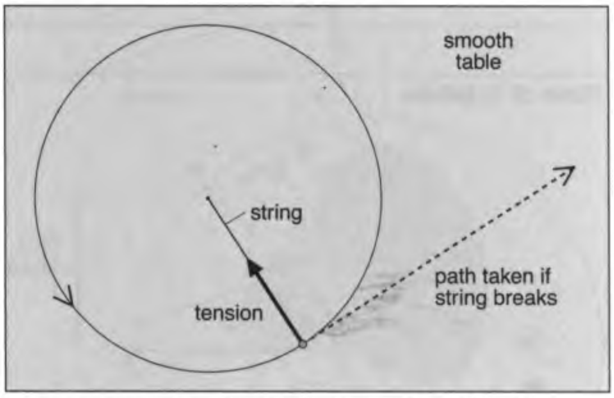


To produce centripetal acceleration, a ***centripetal force*** is needed. It must act towards the centre of the circle. The centripetal force F, mass m, and centripetal acceleration a are linked by the equation $F = ma$. So, using the equations for a in the previous column,

$F = mv\omega$ $F = \frac{mv^2}{r}$ $F = m\omega^2 r$

Note:

- Centripetal force is *not* produced by circular motion. It is the force *needed* for circular motion. Without it, the object would travel in a straight line.



Above, a mass moves in a circle on a smooth table. The tension in the string provides the centripetal force needed. There is no outward 'centrifugal force' on the mass. If the string breaks, the mass travels along a tangent.

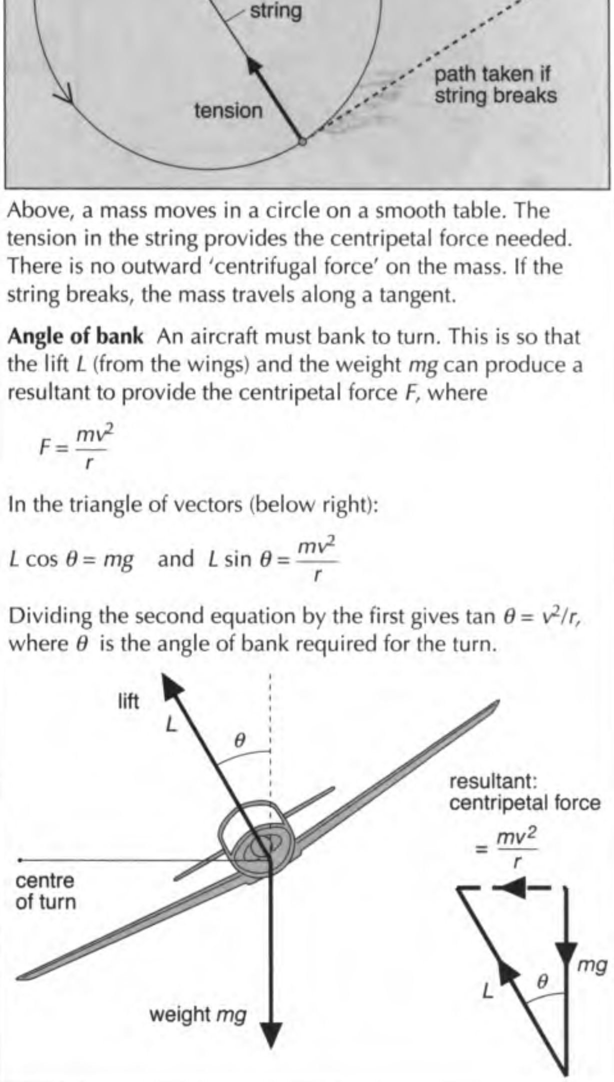
Angle of bank An aircraft must bank to turn. This is so that the lift L (from the wings) and the weight mg can produce a resultant to provide the centripetal force F, where

$$F = \frac{mv^2}{r}$$

In the triangle of vectors (below right):

$L \cos\theta = mg$ and $L \sin\theta = \frac{mv^2}{r}$

Dividing the second equation by the first gives $\tan\theta = v^2/r$, where θ is the angle of bank required for the turn.



B15 Cycles, oscillations and SHM

Periodic motion

This is motion in continually repeating ***cycles***. Here are two examples:

Circular motion Particle P moving at a steady speed in a circle (see the diagram below).

T is the ***period*** (the time for one cycle).
f is the ***frequency*** (the number of cycles per second).
ω is the ***angular velocity*** (measured in rad s^{-1}).

T, f, and ω are linked by the equations below:

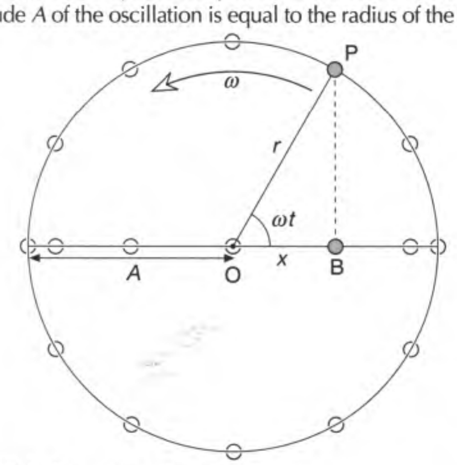
$$T = \frac{1}{f} \qquad \omega = 2\pi f \qquad T = \frac{2\pi}{\omega}$$

(see B11)

Oscillatory motion (e.g. a swinging pendulum)
T, f, and ω are also used when describing oscillatory motion, although ω has no direct physical meaning. They are linked by the same equations as for circular motion.

Linking circular motion and SHM

Below, particle P is moving in a circle with a steady angular velocity ω. Particle B is oscillating about O along the horizontal axis so that it is always vertically above or beneath P. The amplitude A of the oscillation is equal to the radius of the circle, r.



For particle B, $\quad x = r\cos\omega t \qquad (2)$

Using calculus, B's velocity v and acceleration a can be found from the above equation. These are the results:

$$v = -r\omega \sin \omega t \qquad (3)$$

$$a = -r\omega^2 \cos \omega t \qquad (4)$$

From equations (4) and (2), it follows that

$$a = -\omega^2 x \qquad (5)$$

From the diagram, $\sin \omega t = \dfrac{\sqrt{r^2 - x^2}}{r}$

Using equation (3) and remembering that $r = A$, the **amplitude** of the oscillation, the velocity at a distance x from the centre of oscillation can be calculated from

$$v = \omega\sqrt{A^2 - x^2} = 2\pi f \sqrt{(A^2 - x^2)} \qquad (6)$$

$v_{max} = \omega A$ (when x = 0)
$a_{max} = -\omega^2 A$ (when x = A)

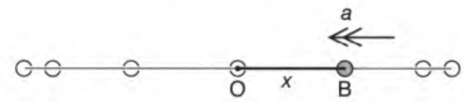
Note:

- Equation (5) has the same form as equation (1). So particle B is moving with SHM.
- The constant in equation (1) is equal to ω^2.
- Using calculus notation, the equation for SHM can be written in the following form:

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

Defining simple harmonic motion

One commonly occurring type of oscillatory motion is called ***simple harmonic motion*** **(SHM)**.

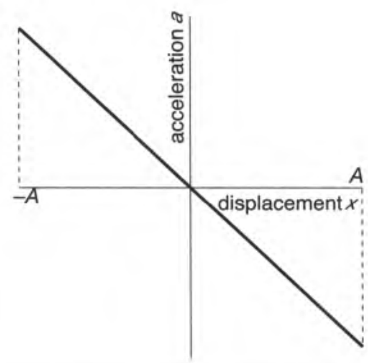


If particle B (above) oscillates about O with SHM, its acceleration is proportional to its displacement from O, and directed towards O.

If x is the displacement, and a is the acceleration (in the x direction), then this can be expressed mathematically:

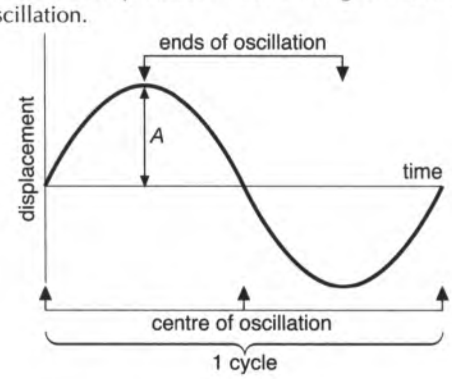
$$a = -\text{(positive constant)}\ x \qquad (1)$$

The minus sign indicates that a is always in the opposite direction to x.



Displacement–time graph for SHM

The following graph shows how the displacement varies with time for one complete oscillation starting from the centre of oscillation.

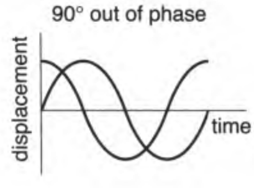


A is the **amplitude** of the oscillation.

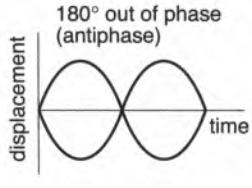
Phase of the oscillation

An oscillation that has the same period but reaches its peak at a different time to that shown above is said to ***oscillate out of phase***. The phase difference between oscillators is quoted as an angle not a time. Two important examples are shown below:

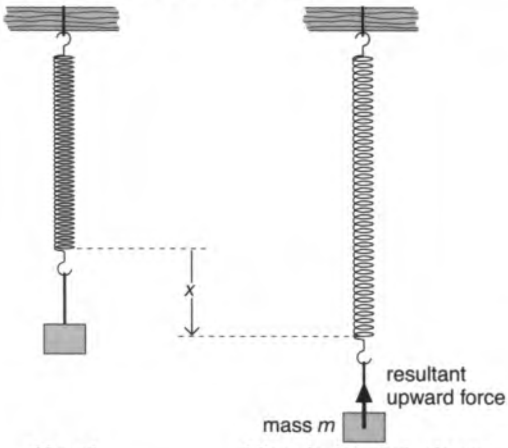
90° out of phase



180° out of phase (antiphase)



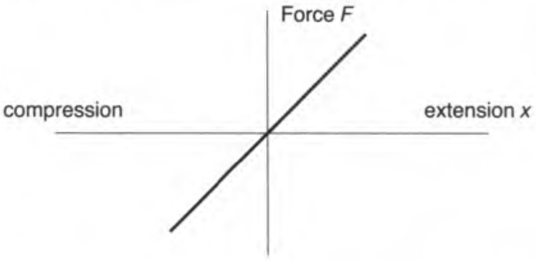
SHM and a mass on a spring



At rest ***After stretch and release***

Above, a mass hangs from a spring. When pulled down and released, the mass makes small, vertical oscillations.

Springs obey Hooke's law. This means that extension x of a spring is directly proportional to the applied force (or load) F.



The graph of applied force against extension is a straight line through the origin.

The gradient of this graph is equal to the ***spring constant*** or ***stiffness of the spring***, k:

$$k = \frac{F}{x}$$

So, if the mass m is pulled down by x and then released,

resultant upward force on mass = kx

But force = mass × acceleration

So acceleration (upwards) $= \frac{kx}{m}$

So acceleration (in x direction) $a = -\frac{kx}{m}$

Comparing this with equation (5) shows that the motion is SHM and that

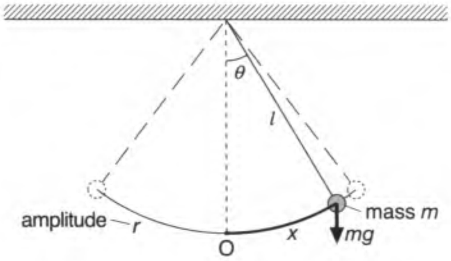
$$\frac{k}{m} = \omega^2$$

As $T = \frac{2\pi}{\omega}$ $\qquad T = 2\pi\sqrt{\frac{m}{k}}$

In any oscillating system to which Hooke's law applies, the motion is SHM.

SHM and the simple pendulum

Provided its swings are small, and air resistance is neglible, a simple pendulum moves with SHM. The following analysis shows why.



The mass m (above) has been displaced by x. It is being pulled towards O by a component of its weight:

force (towards O) = $mg \sin\theta$

But for very small angles $\sin\theta = \frac{x}{l}$ (see B11)

So force (towards O) $= \frac{mgx}{l}$

But force = mass × acceleration

So acceleration (towards O) $= \frac{gx}{l}$

So acceleration (in x direction) $a = \frac{-gx}{l}$

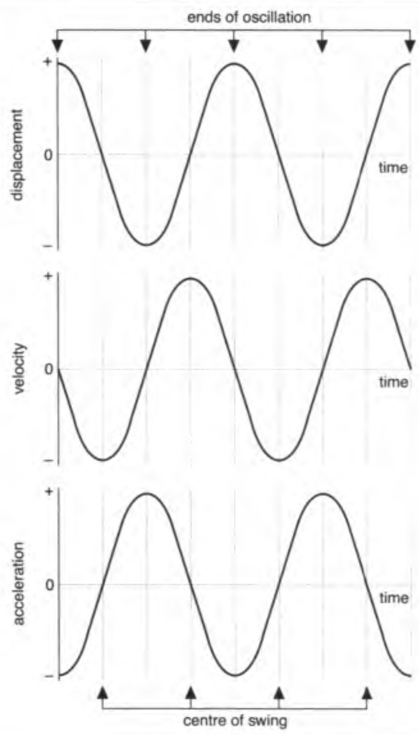
Comparing this with equation (5) shows that the motion is SHM and that

$$\frac{g}{l} = \omega^2$$

As $T = \frac{2\pi}{\omega}$ $\qquad T = 2\pi\sqrt{\frac{l}{g}}$

Note:

- T does not depend on the amplitude (for smaller swings, the period stays the same). This is true for *all* SHM.

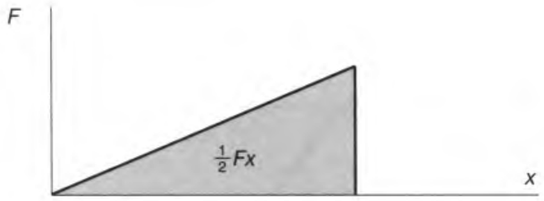


The graphs on the right are for an object moving with SHM: for example, a pendulum making small swings.
At the ends of each oscillation, the velocity is zero. The displacement and acceleration have their peak values, but when one is positive, the other is negative, and vice versa.
At the centre, the velocity has its peak positive or negative value, but the displacement and acceleration are both zero.

B16 Energy changes in oscillators

Mass–spring system

The elastic (or strain energy) stored in a mass–spring system is the work done in stretching the spring. This is the area under the force–displacement graph.

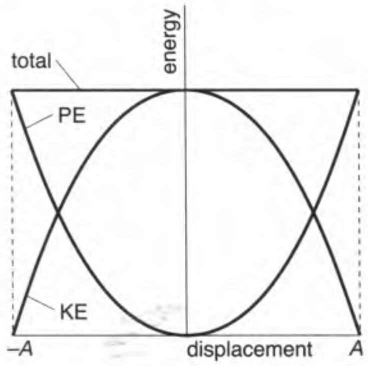


$$\text{elastic energy stored} = \tfrac{1}{2}Fx$$

Since $F = kx$, another useful equation for stored energy is

$$\text{elastic energy stored} = \tfrac{1}{2}kx^2$$

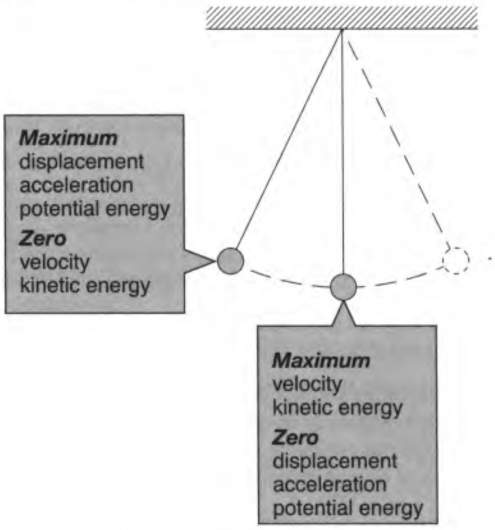
As the spring moves toward the equilibrium position it loses elastic stored energy and gains kinetic energy. But in the absence of any damping the total energy remains constant.



$$\text{maximum elastic stored energy} = \tfrac{1}{2}kA^2$$
$$\text{maximum kinetic energy} = \tfrac{1}{2}m(A\omega)^2$$

Pendulum

For a pendulum the mass loses potential energy (PE) as it swings downwards and gains kinetic energy (KE).



If there is no air resistance the total PE + KE is constant.

total energy = maximum KE = $\frac{1}{2}m(A\omega)^2$

Notice that in all the equations for total energy the total energy is proportional to the *square* of the amplitude (A^2).

Damping

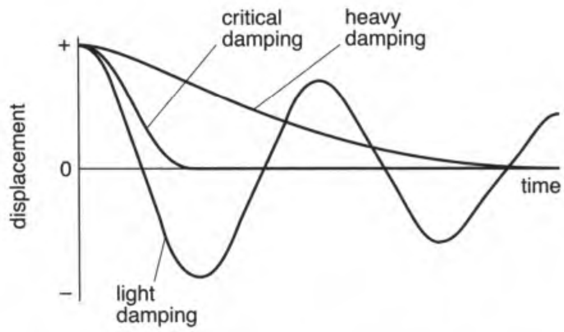
A mass–spring system or a pendulum will not go on swinging for ever. Energy is gradually lost to the surroundings due to air resistance or some other resistive force and the oscillations die away. This effect is called **damping**.

In road vehicles, dampers (wrongly called 'shock absorbers') are fitted to the suspension springs so that unwanted oscillations die away quickly. Some systems (for example moving-coil ammeters and voltmeters) have so much damping that no real oscillations occur. The minimum damping needed for this is called **critical damping**.

The rate at which the amplitude falls depends on the fraction of the existing energy that is lost during each oscillation.

In a **lightly damped** system only a small fraction is lost so that the amplitude of one oscillation is only slightly lower than the one before.

The graphs here are for oscillations with different degrees of damping.



Other uses for dampers include fitting them to buildings in earthquake zones to reduce vibrations so the building can withstand earthquakes. The Millennium Bridge in London has dampers fitted to stop it oscillating from side to side.

B17 Forced oscillations and resonance

Natural frequency

This is the frequency of the oscillation that occurs when the mass of an oscillator (such as a pendulum bob or mass of a mass–spring system) is displaced and then released. The only forces acting are the internal forces of the oscillating system. These oscillations are **free oscillations**.

Forced oscillations

Forced oscillations occur when an external periodic force acts on an object that is free to oscillate.

Examples include:

- engine vibrations making bus windows oscillate
- the spinning drum causing vibrations in a washing machine
- the body of a guitar vibrating when a string is plucked.

The body that is forced to oscillate vibrates at the same frequency as that of the external source that is providing the energy.

The amplitude of the oscillations produced depends on

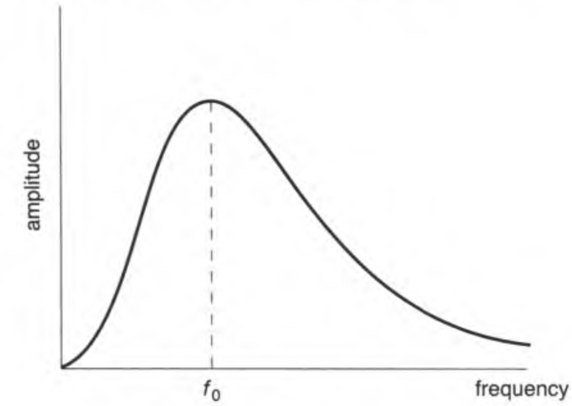
- how close the external frequency is to the natural frequency of the oscillator
- the degree of damping of the oscillating system.

Resonance

Resonance occurs when the frequency of the external source that is driving the oscillation is equal to the natural frequency of the oscillator that is being driven into oscillations.

When resonance occurs the amplitude of the resulting oscillations is a maximum.

The following graph shows how the amplitude of an oscillator with natural frequency f_0 varies with the frequency of the frequency of the source that is driving the oscillations.



Graphs such as these are called ***frequency response graphs***.

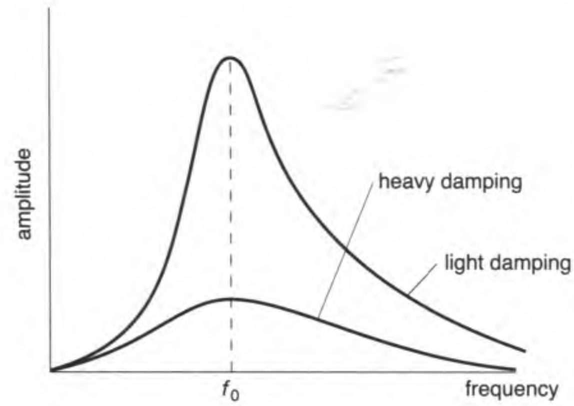
Effect of damping on resonance

When a system is lightly damped, it loses very little energy during an oscillation. If it is being forced to oscillate it will retain most of the energy put into it so that the energy stored builds up and the amplitude becomes very large.

When a system is heavily damped, energy is lost quickly so that the amplitude is lower.

The graph shows the frequency response for lightly and heavily damped oscillators that have the same natural frequency.

Note:
The amplitude of an oscillation stops increasing when the energy put in each cycle is equal to that lost during the cycle.



How Science Works

Electrical resonance

Electrical oscillators made from combinations of capacitors and inductors (coils) can also be forced into oscillations or be made to resonate. This effect is used in the tuner of a radio receiver. The current in the circuit reaches a maximum at the resonant frequency. By tuning the circuit so that the natural frequency matches the frequency of the radio channel, only the frequencies in a small range around this frequency are selected.

The natural frequency of an electrical oscillator depends on the capacitance and inductance of the coil used. By varying the capacitance you can tune in to different channels.

The range of frequencies selected depends on the damping which in turn depends on the resistance in the circuit.

How Science Works

Examples of resonance

Useful resonance

In microwave ovens, the microwaves are produced by a cavity magnetron. There is a resonant frequency that depends on the size of the cavity.

The microwaves are emitted at the resonant frequency and directed into the oven. The frequency is chosen to be one that vibrates the water, fat, and sugar molecules, which heats the food.

Resonance is also useful in musical instruments, for example the air in a clarinet when the reed vibrates.

Resonance to be avoided

Suspension bridges must be designed so that resonance is not caused by the wind vibrating the bridge. The Tacoma Narrows Bridge in Washington State, USA collapsed because of this effect.